Chapter 2
Arithmetic in the Islamic World

1 The Decimal System

Muslim mathematicians were the first people to write numbers the way we do, and, although we are the heirs of the Greeks in geometry, the part of our legacy from the Muslim world is our arithmetic. This is true even if it was Hindu mathematicians in India, probably a few centuries before the rise of Islamic civilization, who began using a numeration system with these two characteristics:

1. The numbers from one to nine are represented by nine digits, all easily made by one or two strokes.
2. The rightmost digit of a numeral counts the number of units, and a unit in any place is ten of that to its right. Thus, the digit in the second place counts the number of tens, that in the third place the number of hundreds (which is ten tens), and so on. A special mark, the zero, is used to indicate that a given place is empty.

These two properties describe our present system of writing whole numbers, and we may summarize the above by saying the Hindus were the first people to use a cipherized, decimal, positional system, "Cipherized" means that the first nine numbers are represented by nine ciphers, or digits, instead of accumulating strokes as the Egyptians and Babylonians did, and "decimal" means that it is base 10. However, the Hindus did not extend this system to represent parts of the unit by decimal fractions, and, since it was the Muslims who first did so, they were the first people to represent numbers as we do. Quite properly, therefore, we call the system "Hindu–Arabic".

As to when the Hindus first began writing whole numbers according to this system, the available evidence shows that the system was not used by the great Indian astronomer Āryabhata (born in A.D. 476), but it was in use by the time of his pupil, Bhaskara I, around the year A.D. 520. (See Van der Waerden and Folkerts for more details.)

News of the discovery spread, for, about 150 years later, Severus Sebokht, a bishop of the Nestorian Church (one of the several Christian faiths existing in the East at the time), wrote from his residence in Keneshra on the upper Euphrates river as follows:

> I will not say anything now of the science of the Hindus, who are not even Syrians, of their subtle discoveries in this science of astronomy, which are even more ingenious than those of the Greeks and Babylonians, and of the fluent method of their calculation, which sur- passes words. I want to say only that it is done with nine signs. If those who believe that they have arrived at the limit of science because they speak Greek had known these things they would perhaps be convinced, even if a bit late, that there are others who know something, not only Greeks but also men of a different language.

It seems, then, that Christian scholars in the Middle East, writing only a few years after the great series of Arab conquests had begun, knew of Hindu numerals through their study of Hindu astronomy. The interest of Christian scholars in astronomy and calculation was, in the main, due to their need to be able to calculate the date of Easter, a problem that stimulated much of the Christian interest in the exact sciences during the early Middle Ages. It is not a trivial problem, because it requires the calculation of the date of the first new moon following the spring equinox. Even the great nineteenth-century mathematician and astronomer C.F. Gauss was not able to solve the problem completely, so it is no wonder that Severus Sebokht was delighted to find in Hindu sources a method of arithmetic that would make calculations easier.

We can perhaps explain the reference to the "nine signs" rather than the ten as follows: the zero (represented by a small circle) was not regarded as one of the digits of the system but simply a mark put in a place when it is empty, i.e., when no digit goes there. The idea that zero represents a number, just as any other digit does, is a modern notion, foreign to medieval thought. This is clearly shown in al-Khalili's auxiliary tables for certain combinations of trigonometric functions depending on two arguments, x and y. In the case of values of x and y that would produce a value outside the domain of the arcos function al-Khālilī writes "0 0," which can only mean 'no value' not zero degrees zero minutes.

With this evidence that the Hindu system of numeration had spread so far by the year A.D. 662, it may be surprising to learn that the earliest Arabic work we know of explaining the Hindu system is one written early in the ninth century whose title may be translated as *The Book of Addition and Subtraction According to the Hindu Calculation*. The author was Muḥammad ibn Mūsā al-Khwārizmī who, since he was born around the year A.D. 780, probably wrote his book after A.D. 800.

We mentioned in Chapter 1 that al-Khwārizmī, who was one of the earliest important Islamic scientists, came from Central Asia and was not an Arab. This was not unusual, for, by and large, in Islamic civilization it was not a man's place (or people) of origin, his native language, or (within limits) his religion that mattered, but his learning and his achievements in his chosen profession.

The question arises, however, where al-Khwārizmī learned of the Hindu arith- metic, given that his home was in a region far from where Bishop Sebokht learned

of Hindu numerals 150 years earlier. In the absence of printed books and modern methods of communication, the penetration of a discovery into a given region by no means implied its spread to adjacent regions. Thus al-Khwārizmī may have learned of Hindu numeration not in his native Khwārizm but in Baghdād, where, around 780, the visit of a delegation of scholars from Sind to the court of the Caliph al-Manṣūr led to the translation of Sanskrit astronomical works. Extant writings of al-Khwārizmī on astronomy show he was much influenced by Hindu methods, and it may be that it was from his study of Hindu astronomy that he learned of Hindu numerals.

Whatever the line of transmission to al-Khwārizmī was, his work helped spread Hindu numeration both in the Islamic world and in the Latin West. Although this work has not survived in the Arabic original (doubtless because it was superseded by superior treatises later on), we possess a Latin translation, made in the twelfth century A.D. From the introduction to this we learn that the work treated all the arithmetic operations and not only addition and subtraction as the title might suggest. Evidently al-Khwārizmī's usage is parallel to the somewhat dated English description of a child who is studying arithmetic as "learning his sums."

2 Kūshyār's Arithmetic

2.1 *Survey of* The Arithmetic

As we have said, al-Khwārizmī's book on arithmetic is no longer extant in Arabic, and one of the earliest works on Hindu numeration whose Arabic text does exist was written by a man named Kūshyār b. Labbān, who was born in the region south of the Caspian Sea some 150 years after al-Khwārizmī wrote his book on arithmetic. Although Kūshyār was an accomplished astronomer, we know very little about his life, but despite this personal obscurity his works exerted some influence, and his treatise on arithmetic, whose title means *Principles of Hindu Reckoning*, became one of the main arithmetic textbooks in the Islamic world.

Kūshyār's concise treatise is a carefully written introduction to arithmetic, divided into two main parts. The first contains, after a brief introduction, nine sections on decimal arithmetic, beginning with "On Understanding the Forms of the Nine Numerals." In this the nine numerals are given in a form standard in the east, namely:

$$\text{١ ٢ ٣ ٤ ٥ ٦ ٧ ٨ ٩}$$

and the place-value system is explained. Zero is introduced as the symbol to be placed in a position "where there is no number." The Arabic word for zero, "ṣifr," comes from the verb "ṣafira" which means "to be empty or void" and it is the source, via French and Spanish, of our word "cipher". It is even the source, via

Italian, of our word "zero." The sections following the first are "On Addition," "On Subtraction"—with halving treated in a special subsection, "On Multiplication," "On the Results of Multiplication," "On Division," "On the results of Division," "On the Square Root," and "On Checking," through casting out nines.

The sixteen sections of the second part contain an explanation of the arithmetic of a base-60 positional system, but the book concludes with a section that tells how to find the cube root of a number in the decimal system. The base-60 system, which is now called a *sexagesimal* system, was important to astronomers because angles were measured, and trigonometric functions were tabulated, according to this system, and because its unified treatment of whole numbers and fractions made calculations so much simpler. We shall say more of this later.

As we follow Kūshyār's explanation of the decimal system it is well to bear in mind that he was explaining arithmetic to people who would be computing not with pen or paper but with a stick (or a finger) on a shallow tray covered with fine sand, which we shall refer to as a "dust board." Because small boards are more convenient to carry around than large ones, it is desirable to have arithmetic algorithms that do not require writing down several rows of numbers. On the other hand, it is easy to erase on a dust board, so algorithms that require considerable erasing pose no problem, and we shall see how the algorithms for addition, subtraction, multiplication, division, and extracting square roots were designed with this feature of the dust board in mind.

In the text of his book Kūshyār writes out, in words, all the names of the numbers, and it is only when he is actually exhibiting what is written down on the dustboard that he uses the Hindu-Arabic ciphers. A reason for this may be that explanations were considered as text, and therefore written out in words, like any other text. The examples of what was written on the dust board, however, may have been viewed as illustrations, much like a diagram in a geometrical argument, and they were there to show what the calculator would actually see on the dust board.

2.2 Addition

As Kūshyār explains this, the numbers to be added are written in two rows, one above the other, so that places of the same value are in the same column. He gives the example of adding 839 to 5625 and, unlike our method, begins his addition by adding from the highest place common to both numbers, in this case the hundreds' place, down. At each stage the answer obtained so far replaces part of the number on the top. Figure 1 illustrates his steps, beginning with 56 + 8 = 64, and an arrow (→) shows that the display on the right replaces, on the dust board, that on the left. Thus, at any time, there are only two numbers on the dust board, arranged in columns, and, in the end, the answer has replaced the number on the top. Unlike our method, the method Kūshyār explains obtains the leftmost digit of the answer first.
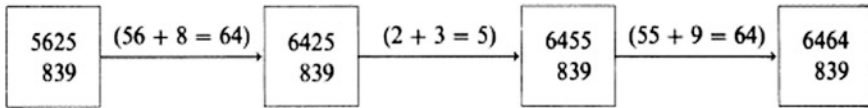
5625	(56 + 8 = 64)	6425	(2 + 3 = 5)	6455	(55 + 9 = 64)	6464
839		839		839		839

Fig. 1

2.3   Subtraction

Again Kūshyār explains the method by the same numbers, subtracting 839 from 5625, and again he works from left to right. He explains that since 8 cannot be subtracted from 6 it must be subtracted from 56 to produce 48, so the 56 of 5625 is erased and replaced by 48. Thus, working from place to place, Kūshyār obtains the answer, 4786 (Fig. 2). There is no "borrowing" in Kūshyār's procedure. He simply notices that, for example, in the last step, since we cannot subtract 9 from 5 we must subtract it from 95. Just as with addition Kūshyār works from the higher places to the lower, and at each stage the partial answer appears as part of the number on top.

His treatment of halving, which he considers to be a variant of subtraction, sheds light on his treatment of fractions. He begins with 5625 (as usual), but this time he starts on the right (Fig. 3). He says to set down 5625 and then take half of five, which is two and a half. "Put two in the place of the five and put the $\frac{1}{2}$ under it, thirty."

He is using here, for fractions, the sexagesimal system, which goes back to the Babylonians and uses the principle of place value to represent fractions in terms of multiples of the subunits $1/60$, $1/60^2 = 1/3600$, etc. He explains the system more fully in the second part of the treatise, and here he contents himself with using his readers' familiarity both with the local monetary system in which a *dirham* contained 60 *fulūs*, and with degrees, in which 1 degree contains 60 minutes. Thus he tells his reader, in effect, "If you wish to think of 5625 as *dirhams* (degrees), then
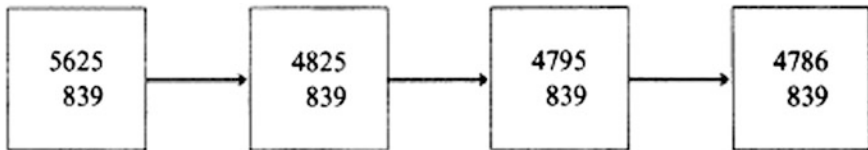
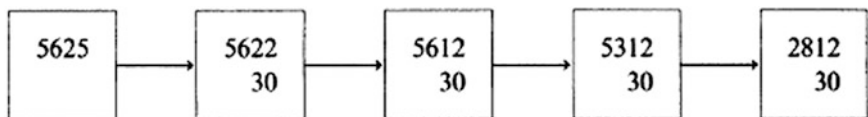| 5625	4825	4795	4786
839	839	839	839

Fig. 2

5625	5622	5612	5312	2812
	30	30	30	30

Fig. 3

think of $\frac{5}{2}$ *dirhams* as 2 *dirhams* and 30 *fulūs* (2 degrees and 30 minutes)." The next
two steps of his calculation, as shown in Fig. 3, are to halve the 2 in the 10's place
and then the 6 in the 100's place, and now he must take half of the 5 in the 1000's
place. Kūshyār values the 5 in terms of the preceding place, and so looks on it as 50
hundreds. Its half is thus 25 hundreds, and so in the last step he adds the 2500 to the
half of 625 to obtain the answer shown in Fig. 3.

2.4 Multiplication

The algorithm for multiplication shows a thorough understanding of the rule for
multiplying powers of 10, for to multiply 243 by 325 Kūshyār requires his reader to
arrange the numerals so the 3 of 325 is directly above the 3 of 243 (Fig. 4). The
total array occupies five columns, because hundreds multiplied by hundreds yields
tens of thousands $((N \cdot 100) \cdot (M \cdot 100) = N \cdot M \cdot 10,000)$. Since $3 \cdot 2 = 6$, he
places the 6 directly above the 2, i.e., in the ten thousands' place, and he remarks
that had the product produced a two-digit number (e.g., had it been $4 \cdot 3 = 12$), the
tens' digit of the product would be placed in the column to the left of the 2. This is
illustrated at the next step where, since $3 \cdot 4 = 12$, he places the 2 of the 12 directly
above the 4 and adds the 1 to the 6 to get 72. Finally, the top 3 is replaced by the
$9 = 3 \cdot 3$, since he no longer needs to multiply by it.

Now we will be multiplying 243 by the upper 2, and since this counts "tens" and
not "hundreds" we must, if we are to continue adding the answers to the top row in
the columns above the bottom numerals, shift 243 one place to the right, since the
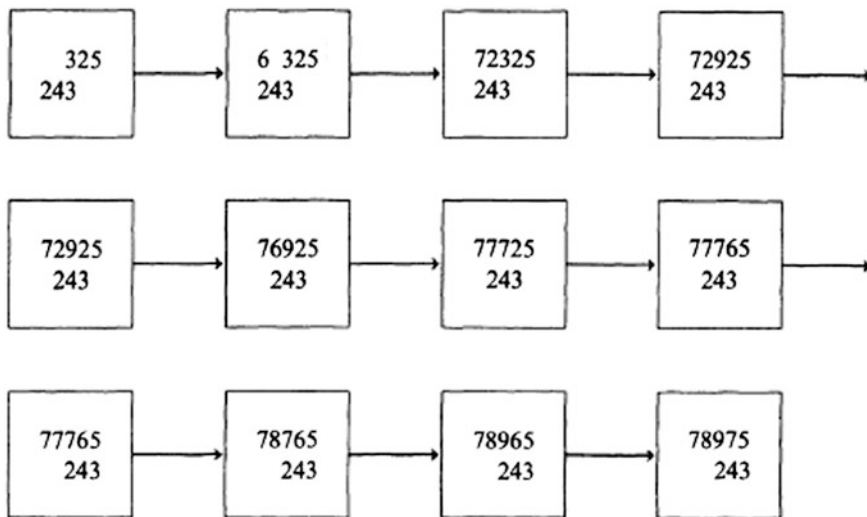

Fig. 4

powers of 10 represented by the answers will be one less. Thus, we begin in the second row of Fig. 4, and as before, the last digit of the lower number (3) stands under the current multiplier (2). Then, since 2 · 2 = 4, we add the 4 to 72 to get 76, and the remaining steps of Row 2 will be clear to the reader who has followed those of Row 1. Again, a shift to the right automatically lines up the figures so that the answers are put in the correct place. We thus have Row 3 of Fig. 4, and the only thing to be careful of is that on the last multiplication of a given sequence (for example the final "5 · 3 = 15") one does not add the final digit of the product to the last multiplier (in this case, 5), but, instead, uses it to replace the multiplier.

2.5 *Division*

This operation offers Kūshyār no more trouble than multiplication, as the division of 5625 by 243 shows. In Fig. 5 the divisor, 243, is written at the bottom and the dividend, 5625, is written above that, its highest place written above the highest place of the divisor. The first digit of the quotient, 2, is obtained by an estimate and is written in the column above the last digit of the divisor, 243. In Fig. 5 the first three steps show the process of subtracting 2 · 243 = 486 from 562. In this case, the "2," since it is written above the tens' place of the dividend, means 20, and the positioning automatically puts it in the right column. This process is shown in the first four boxes of Fig. 5, and the fourth box says that 5625 – 20 · 243 = 765 Now Kūshyār moves the divisor one column to the right, so that the next digit of the quotient will be correctly aligned. The second row calculates 765 – 3 · 243 = 36, and so, calculating digit by digit, beginning with the one in the highest place, Kūshyār obtains the quotient (23) and remainder (36).

This answer, 23 + 36/243, is correct but raises the further question, "How big is the fraction 36/243?" After all, an astronomer doing calculations with angles, or a judge dividing up a sum of money as an inheritance, needs the answer in a usable form. Thus, a standard chapter in many arithmetics is one explaining how to express a fraction *a/b* in terms of some other subunits l/*c*, where *c* is a number appropriate to
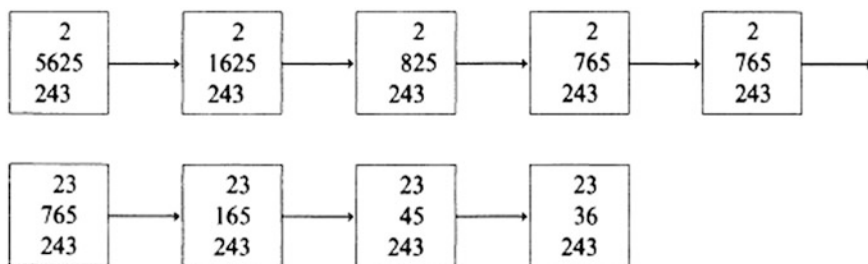

Fig. 5

what is being measured. For example, if we were measuring lengths in feet and inches we would take $c = 12$, but Kūshyār proceeds to solve this problem for $c = 60$.

Of course, if $36/243 = n/60$, then $n = 36 \cdot 60/243$, and this division produces a quotient of 8 and a remainder of 216. So, if we think of the first remainder as being the *dirhams* left after the division of 5625 *dirhams* among 243 people, then each person's share would be 23 *dirhams* 8 *fulūs*, with 216 *fulūs* left over. Or, we could think of it as the division of an angle into 243 equal parts, so that each part would be 23°8′ with 216′ left over. This operation of multiplying a fraction, as $36/243$, by 60 the Islamic authors called "raising," and it was used to obtain base-60 expansions of the fractional parts in a division. It is the analog of what we do to convert a fraction to percent.

3 The Arithmetic of Common Fractions

Common fractions appear in Kūshyār's work, as we have seen, in the context of division, where the division of 5,625 by 243 leads to a quotient of 23 and a remainder, 36, being written above the divisor. And Kūshyār refers to the quotient as 23 and 36 parts of 243 [parts] making up the unit. However, common fractions arise in practical problems of science, commerce, finance, law, etc. and practitioners (and their teachers) developed a number of ways of relating such fractions to simpler common fractions, and such fractions were discussed in numerous works of arithmetic aimed at a wide audience. One such book, which was much studied in the Maghrib, was the *Book of Demonstration and Reminder*, written by the twelfth century the mathematician Abū Bakr Muḥammad al-Ḥaṣṣār.[1] So far as is known it was he who introduced the (now usual) notation of a horizontal bar to separate the numerator of a common fraction from its denominator, a notation adopted by all subsequent authors of arithmetic texts in the Maghrib. In his writings we also meet for the first time a fivefold classification of fractions, which we shall discuss below.

The works on arithmetic were teaching texts, and teachers have always found it useful to systematize knowledge in such a way that students can gain a view of the whole subject. Thus, al-Ḥaṣṣār describes five kinds of fractions, the most basic being *simple* fractions, i.e., the nine unit fractions ½, ⅓, …, ¹⁄₁₀. Then there were fractions *related* to others, an example being fractions of the form $\frac{c}{d} + \frac{a}{b} \cdot \frac{1}{d}$, which were written

$$\frac{a\ c}{b\ d}.$$

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[1]His other work, also on arithmetic, was titled *Al-Kāmil* (The Complete/Perfect).

For example, the related fraction

$$\frac{5\ 4}{8\ 9}$$

would be read, "four ninths and five-eighths of a ninth."

Writers could—and did—extend that notation to relate three or more fractions in the same manner, and the related fraction

$$\frac{1\ 5\ 4}{2\ 8\ 9}$$

would be read four ninths and five times an eighth of a ninth and one half of an eighth of a ninth.

The exact dates of al-Ḥaṣṣār are not known, but it was likely about a century after he was born that the mathematician Aḥmad Ibn al-Bannā' was born in Marrakesh in 1256. During his lifetime he achieved sufficient fame as a skilled mathematician and astronomer/astrologer to be invited to the capital, Fās (Fez), a number of times by the Merinid sultans. (He was also famous as a mystic and magician, who could work wonders!) He died in 1321, leaving fourteen books on mathematics alone and a number of students who continued his work.

The two mathematical works for which Ibn al-Bannā' is best known today are *A Summary Account of the Operations of Computation* and *Raising the Veil on the Various Procedures of Calculation*. The first was evidently too much of a summary for some readers, since the second is a commentary on the first, expanding on the material in it. The great Tunisian historian, Ibn Khaldūn, who was born very shortly after Ibn al-Bannā''s death and was well acquainted with mathematics, said of Ibn al-Bannā''s commentary, "It is an important work. We have heard our teachers praise it, and it deserves that."

In his *Summary* Ibn al-Bannā' follows the classification we find in al-Ḥaṣṣār and explains that to find the numerator of a related fraction

$$\frac{a\ c}{b\ d}$$

one multiplies the number written above the first denominator[2] by the second denominator and adds the product to the number above that denominator.[3] Thus, in the case above, the numerator would be $c \cdot b + a$. He then states the general rule: "One multiplies what is above the first denominator by the following denominators, that which is above the second by the denominators following it, and so on to the end of the line. Then one adds these products." So, in the numerical example above,

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[2]Since Arabic is read from right to left, 'first' denominator means the rightmost.

[3]We have relied on Souissi (1969).

$$\frac{1\,5\,4}{2\,8\,9}$$

the numerator would be $4 \cdot 8 \cdot 2 + 5 \cdot 2 + 1$, i.e., 75. The denominator, of course, would be $9 \cdot 8 \cdot 2$, i.e., 144.

He then proceeds to give rules for dealing with fractions mixed with whole numbers. For example, in the case of an expression such as

$$\frac{a\,c}{b\,d}n$$

where n is a whole number, his rule amounts to calculating the numerator as $n \cdot (b \cdot d) + (a + c \cdot b)$, that is, to say one interprets the expression as meaning the <u>sum</u> of n and the fraction

$$\frac{a\,c}{b\,d}.$$

But, if the whole number, n, is found on the other side of the related fraction the numerator is the <u>product</u> of n and that of the related fraction.

Then there were *different* fractions, which were simply sums of fractions, $\frac{a}{b} + \frac{c}{d}$, *partitioned* fractions, which was the term for products, $\frac{a}{b} \cdot \frac{c}{d}$, and, finally, fractions *separated by a sign of subtraction*. (That sign was the Arabic word *illā* ("except") prefixed to the fraction being subtracted.

In the case of *different* fractions he obtains the numerator as we do, by multiplying each numerator by the other denominator and adding the two results. For *partitioned* fractions Ibn al-Bannā' writes, 'One multiplies the numbers written above the line by each other,' a clear reference to the horizontal fraction bar. For the case of *subtraction* he says that one proceeds as for addition but then subtracts the smaller product from the larger.

The exposition of the theory and practice of calculating with fractions was, as the above exposition hints, one that received considerable attention. Thus, Aḥmad b. Munʿim, who died in Marrakesh in 1228 (and of whom the reader will learn much more later), devoted nearly half of his large work, *The Laws of Calculation*, to the topic of fractions.

Some one hundred fifty years after the death of Ibn al-Bannā', an Andalusian mathematician, ʿAlī b. Muḥammad al-Qalaṣādī, who died in Tunis in 1486 (only 6 years before the voyage of Columbus), wrote a work whose title is an obvious reference to the commentary of Ibn al-Bannā', namely *Removing the Veil from the Science of Calculation*. As did Ibn al-Bannā', al-Qalaṣādī also wrote on religious topics and literature, but it is his commentary on Ibn al-Bannā''s *Summary Account* that concerns us here.

In Part I of his book, in his discussion of multiplication, he gives some rules for multiplication which make one realize why, in Kūshyār's *Hindu Reckoning*, for

example, one finds halving and doubling treated as separate topics.[4] Thus, al-Qalaṣādī says

"To multiply a number by three, add it to its double,"

and

"To multiply a number by six, add it to half of its product by ten,"

"To multiply a number by seven, put a zero to its right and subtract its triple from its product by ten," i.e. $7a = 10a - 3a$, the product $3a$ being calculated as above.

But he goes on with more complicated rules, such as multiplying a number by twelve by placing the number directly below itself and then placing the number again below the two first, but so that the units place of the lowest line is lined up with the tens places of the two lines above. Add these three numbers and the result will be the answer. The calculation of 147×12 would look like this:

$$147$$
$$147$$
$$147$$
$$1764$$

Al-Qalaṣādī explains division as decomposing the dividend into parts equal to the divisor, and then, in Part I, Chapter 5, he applies it to the problem of factoring numbers. He first gives the usual test for seeing if a number is divisible by 9, a test he calls 'reducing the number by 9.' He adds that if an even number is divisible by 9 then it is also divisible by 6 and 3. But if 3 or 6 remain when one reduces a number by 9, as with such numbers as 48 and 78, then it is only divisible by 3 and 6.

If none of this works, reduce it by 8, and his procedure makes it clear that he knows multiples of 1,000 are divisible by 8. So he only needs to reduce a three-digit number by 8. For example, since 174 leaves a remainder of 6 when divided by 8 so does is 3174.

For reduction by 7 his process mirrors long division, although his description of it is interesting. He says, "Think of the leftmost digit as tens and add it to the digit to its right, considered as being units. Reduce the sum by seven. Then add the remainder [after this reduction], thought of, again, as tens, to the next digit to the right, and continue the reduction in this way."

He applies this procedure to 5236, where the calculations go $5236 \rightarrow \underline{3}36 \rightarrow \underline{5}6$, where we have underlined the remainders after the successive division by 7. One concludes 5236 is divisible by 7.

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[4]Interestingly, however, according to Djebbar (1992) the treatment of doubling and halving as separate topics in Arabic arithmetic was, after al-Ḥaṣṣār, dropped in the Maghrib and the topics were dealt with as special cases of multiplication and division.

All of this comes together in a problem in which he shows the reader how to express a fraction as a related fraction. His rule is to express the denominator as a product of factors and write those factors in a row (in descending order from right to left) and place a line over the factors. The divide the numerator by these factors, one after another. "You will obtain the result sought." As with so much mathematical instruction, this somewhat cryptic rule becomes clear with an example in which al-Qalaṣādī shows how to express 75/144 as a related fraction.

"And if someone says to you, 'Denominate seventy-five according to one hundred forty-four,' you decompose the denominator into nine, eight, and two and divide the numerator first by two, obtaining 37 with a remainder of one, which you put above the two. Then divide the quotient by eight to get four [with a reminder of five, which you put above the eight].[5] Put the four above the nine. The result will be four-ninths and five-eighths of a ninth and a half of an eighth of a ninth. Write it as follows:

$$\frac{1\ 5\ 4}{2\ 8\ 9}\text{"}.$$

And, in following through his reasoning, one can see how this notation for fractions, so different from ours, might have arisen.

In Part II, Chapter 4 al-Qalaṣādī treats the division of one expression involving connected fractions by another of the same type. His rule is to form the product of the numerator of each of the two fractions by the factors [of the denominator] of the other, and then divide the product of the dividend by that of the divisor, after having decomposed this (latter product) into its factors. He then gives the following example: Divide ¾ and ⁵⁄₇ of ¼ by ⅖ and ⁶⁄₇ of ⅕.

Thus

$$\frac{5\ 3}{7\ 4}$$

is to be divided by

$$\frac{6\ 2}{7\ 5}.$$

The numerator of the dividend is 26 and the factors of its denominator are 7 and 4. The numerator of the divisor is 20 and the factors of its divisor are 7 and 5. One must then form the products $26 \times 7 \times 5$ (= 910) and divide it by the products $20 \times 7 \times 4$ (= 560). Decompose the latter number into its factors, which are 10, 8, and 7 and divide 910 by these factors. He expresses the answer as

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[5]At some point in the history of this text, the bracketed phrase I have inserted, clearly essential, was left out.

$$\frac{0\ 2\ 6}{7\ 8\ 10}1$$

in other words, one and six-tenths and two-eighths of a tenth.

One is reminded of Kūshyār's procedure in the example we gave earlier of calculating in the sexagesimal system. In that case, it was converting the sexagesimal numbers to whole numbers expressed in the decimal, positional system then doing the calculation in that system, and finally converting the answer back to sexagesimals. Here, one converts the related fractions to ordinary common fractions, then calculates the quotient with them, and finally converts the result back to related fractions.

4   The Discovery of Decimal Fractions

Today we use not sexagesimal but decimal fractions to represent the fraction remaining after a division, and it now appears these were a contribution of the Islamic world. Evidence for this claim is contained in *The Book of Chapters on Hindu Arithmetic*, written in Damascus in the years A.D. 952–953 by Abū al-Ḥasan al-Uqlīdisī. The name "al-Uqlīdisī" indicates that the author earned his living copying manuscripts of Euclid ("Uqlīdis" in Arabic), but beyond this we know nothing of the life of a man who seems to have been the first to use decimal fractions, complete with the decimal point, and therefore the first to write numbers as we do. Since al-Uqlīdisī specifically states in the preface to his book that he has taken great pains to include the best methods of all previous writers on the subject, it is hard to be sure that the decimal fractions were his own discovery, but their complete absence in Indian sources makes it fairly certain that they were a discovery of Islamic scientists.

Al-Uqlīdisī is also proud of the fact that he has collected ways of performing on paper, with ink, algorithms usually performed by arithmeticians on the dust board, and in his *Book of Chapters* he gives the following reasons for abandoning the dust board in favor of pen and paper.

> Many a man hates to show the dust board in his hands when he needs to use this art of calculation (Hindu arithmetic) for fear of misunderstanding from those present who see it in his hands. It is unbecoming him since it is seen in the hands of the good-for-nothings earning their living by astrology in the streets.

It seems that the street astrologers could be recognized by their use of the dust board, and al-Uqlīdisī urges the use of pen and paper to escape being taken for a mendicant fortune teller.

Al-Uqlīdisī's text contains four parts, of which the first two deal with the elementary and advanced parts of Hindu arithmetic, and it is in the second part where decimal fractions first appear. This is in the section on doubling and halving numbers, where he introduces them as one of the three ways of halving an odd

number. The first way is the one described by Kūshyār who, to halve 5625, considered it as degrees or *dirhams* and wrote the result as in Fig. 3, where the lower 30 could be interpreted as *fulūs* or minutes. The second way is one al-Uqlīdisī calls numerical, and describes as follows:

> … halving one in any place is five (in the place) before it, and this necessitates that when we halve an odd number we make half of the unit five before it and we put over the units' place a mark by which we distinguish the place. So the value of the unit's place is tens to that before it. Now the five may be halved just as whole numbers are halved, and the value of the units' place in the second halving becomes hundreds and this may continue indefinitely.

When al-Uqlīdisī writes of places in a numeral being "before" other places he is referring to the direction of Arabic writing, which is from right to left. Thus in 175 the 5 would be before the 7. As an example of what he has explained al-Uqlīdisī gives the results of halving 19 five times as 059375, where, he says, "the place of the units is hundred-thousands to what is in front of it." Figure 6 shows the Arabic text of al-Uqlīdisī's work and the use of the decimal point in the form of the short vertical mark pointing out the unit's place. The decimal mark is clearly visible above the '2' in the middle of line 10 of the text in Fig. 6 and over the '9' at the left end of that line. Using the forms of the numerals given earlier, the reader will have no trouble identifying the various numerals in that figure.

From a purely mathematical point of view it is especially satisfying to see decimal fractions, complete with decimal point, explained by reasoning by analogy from established procedures. The usual procedure for halving an even number, such as 34, was to begin by halving the units, so 34 → 32. Then, as a writer like Kushyar would have put it, "The three is tens of the two to the right of it, so its half is fifteen. We add the five to the two, which is its units, and it becomes seven. So the result is 17." The principle used here was that half a unit in one place (tens, hundreds, etc.) was five in the place to the right. Al-Uqlīdisī observed that the same principle could be applied to halving a number with an odd digit in the unit's place, and out of such a simple observation came a very useful mathematical tool.

A little later, al-Uqlīdisī again uses decimal fractions, this time to increase 135 by its tenth, then the result by its tenth, etc. five times. Thus he sets out as in Fig. 7 to calculate $135\left(1+\frac{1}{10}\right)^5$. He writes 135 and below it 135 again, but moved one place to the right. This will be $\left(\frac{1}{10}\right) \cdot 135$, so he adds it to 135. In the sum $135 + \left(\frac{1}{10}\right) \cdot 135$, he marks the unit's place with a short vertical line above it. When he has shifted and added four more times, the result will be the desired quantity. (He mentions that the value of the lowest place is hundred-thousandths.)

He gives an alternative to this method as follows (where we use the decimal point for al-Uqlīdisī's vertical line):

$$135 \cdot \left(1 + \frac{1}{10}\right) = \frac{(135 \cdot 11)}{10} = 148.5$$

Fig. 6

135 $\longrightarrow$ 1485 $\longrightarrow$ 16335 $\longrightarrow$ 179685 $\longrightarrow$ 1976535 $\longrightarrow$ 21741885
135   1485   16335   179685   1976535

Fig. 7

and

$$148.5 \cdot \left(1 + \frac{1}{10}\right) = \frac{(148.5 \cdot 11)}{10} = 148.5\left(\frac{11}{10}\right) + 0.5 \cdot \left(\frac{11}{10}\right)$$
$$= 162.8 + 0.55 = 163.35,$$

which shows that al-Uqlīdisī not only added decimal fractions but multiplied them by whole numbers as well, even though his method of multiplication unnecessarily splits the number into its whole and fractional parts.

Less than half a century later another Muslim author, Abū Manṣūr al-Baghdādī, used decimal fractions—also in a problem on computing tenths. He represented what al-Uqlīdisī would write as $17\overset{\prime}{2}8$ by 08 02 17, but each pair written above the previous one, in strict analogy to Kūshyār's notation for sexagesimal fractions.

Al-Uqlīdisī's use of decimal fractions is something of an ad hoc device, unsystematized and unnamed. Two centuries later, however, one finds in the writings of al-Samaw'al, whose work we discuss in the chapter on algebra, the use of decimal fractions in the context of division and root extraction. In a treatise of 1172, al-Samaw'al introduces them carefully, as part of a general method of approximating numbers as closely as one likes. Thus al-Samaw'al uses decimal fractions within a theory rather than as an ad hoc device, although he still has no name for them and his notation is inferior to that of al-Uqlīdisī. The reader will find the details in Rashed.

It is in the early fifteenth century that decimal fractions receive both a name and a systematic exposition. By then Jamshīd al-Kāshī displays a thorough command of the arithmetic of decimal fractions, for example, multiplying them just as we do today. It is also in the fifteenth century that a Byzantine arithmetic textbook describes as "Turkish", i.e., from the Islamic world, the method of representing $153\frac{1}{2}$ and $16\frac{1}{4}$ by 153|5 and 16|25 and their product by 2494|375. (See Hunger and Vogel.)

It was not until over a century later that the European writers began using decimal fractions. An able publicist for the idea was the Flemish engineer Simon Stevin, whose book *The Tenth* was published in 1585. However, his awkward notation was nowhere near so good as al-Uqlīdisī's, and it was left to the Scot, John Napier, to reinvent the decimal point and use decimal fractions in his table of logarithms, another invention of his.

5   Muslim Sexagesimal Arithmetic

5.1   *History of Sexagesimals*

Although the student may think it strange that it took almost 500 years (from the tenth to the fifteenth centuries) for decimal fractions to develop, it must be remembered that Muslim scientists, from the ninth century onwards, already

possessed a completely satisfactory place-value system to express both whole numbers and fractions. It was not decimal, however, but the *sexagesimal* system we have already referred to, in which the base is 60, and it arose out of the fusion of two ancient numeration systems.

The first of these is one used by the Babylonians around 2000 B.C in Mesopotamia. As we know it from the many surviving cuneiform texts, it was a positional system, in which the successive places of a numeral represent the successive powers (positive and negative) of the base, 60, so it treated whole numbers and fractions in a unified manner. However, the Babylonians did not use single ciphers for the fifty-nine digits from 1 to 59, but formed them by repeating the wedges for 1 (𐏑) and 10 (𐏒). Thus, the Babylonians would represent the integers 3, 25, 133 and 3753 as

$$\text{𐏒, 𐏒𐏑, 𐏑𐏒} \quad (2 \cdot 60 + 13), \quad \text{𐏑 𐏒𐏑} \quad (1 \cdot 60^2 + 2 \cdot 60 + 33).$$

In addition, they extended the system to include fractions. Thus since $\frac{1}{2} = \frac{30}{60}$ they, would write $\frac{1}{2}$ as 30 also 𐏒𐏒𐏒 and since

$$\frac{7}{360} = \frac{70}{3600} = \frac{60}{3600} + \frac{10}{3600} = \frac{1}{60} + \frac{10}{60^2}$$

it would be written as 𐏑𐏒.

There was always a possibility of misunderstanding in using this system, for there was no special mark to indicate the units (that is, there was no sign like our decimal point and no custom of writing final zeros in an integer), and therefore, the magnitude of the number was determined only up to a factor of some power of 60. Thus, although 𐏑𐏒 could represent $1\frac{1}{3}$, it could also represent 80. One step towards clarification was taken late in the fourth century B.C., at the time when Babylon was ruled by the successors of Alexander. At that time, scribes in Babylon began to write numbers more frequently with a special symbol to indicate zeros within the numeral, so it was possible to write 71 in such a way as to distinguish it unambiguously from 3611 (71 being written as 𐏑𐏒 and 3611 as 𐏑𐏒𐏑).

These imperfections, however, are relatively minor and seemed not to have caused much difficulty for the Babylonians. Much more important is the existence, two millennia before our era, of a numeration system so well suited for complex calculations that Greek astronomers, at some time during or after the second century B.C., adopted it for their calculations. Thus, the astronomer Ptolemy used it in the mid-second century A.D. in his Greek astronomical handbook *The Almagest*.

The Hellenistic Greeks' adoption of the system was, however, rather an instance of grafting than of transplanting; for, while they used it with a different notation to represent the fractional part of a number, they retained their own method of representing the integral part. This method is an example of the second ancient system we referred to earlier, in which 27 letters of an alphabet are used to represent the numbers 1,…, 9; 10, 20,…, 90; 100, 200,…, 900. In the case of the Greeks, they

used 27 letters of an archaic form of the Greek alphabet, according to the scheme below:

Α	Β	Γ	Δ	Є	Ϛ	Ζ	Η	Θ
1	2	3	4	5	6	7	8	9
Ι	Κ	Λ	Μ	Ν	Ξ	Ο	Π	Ϙ
10	20	30	40	50	60	70	80	90
Ρ	Σ	Τ	Υ	Φ	Χ	Ψ	Ω	ϡ
100	200	300	400	500	600	700	800	900

This ancient alphabet stems from that of the Phoenecians, a Semitic people to whom we owe the inventions of the alphabet and of money. The alphabetic system of numeration seems to have been common to many of the peoples of the Mediterranean. Thus, it was used not only by the Greeks and Arabs, but also by the Hebrews and others.

In this system the numbers up to 999 would be represented by a string of letters, so that, in the case of the Greeks 48 and 377 would be written MH and TOZ. We need not go into the special devices that were used to represent numbers larger than 999, for it is the fractions that interest us now. A Greek astronomer, knowing the Babylonian system, evidently saw the possibility of substituting letters of the alphabet for the groups of wedges the Babylonians used for digits. Thus $12\frac{1}{3}$ would be written *IB K*, to signify $(10 + 2) + \frac{20}{60}$.

The Greeks, however, adapted the Babylonian place-value system only for fractions, so they wrote *PMB IB* for $142\frac{1}{5}$ rather than the more consistent *B KB IB* (i.e. $2 \cdot 60 + 22 + \frac{12}{60}$). The only improvement the Greek system displayed was a slight cipherization for the digits, so that whereas the Babylonian would have to write ≪ for $\frac{1}{3}$, the Greek could simply write *K*.

The real transplant of the Babylonian system was done by Islamic mathematicians, in a system that was so widely used by astronomers that it simply became known as "the astronomers' arithmetic." In it, the 28 letters of the Arabic alphabet were used in an order quite different from their order in the alphabet as it was (and is) written. If we transcribe these letters according to the system in Haywood and Nahmad then the correspondence between letters of the Arabic alphabet and numerals is that shown in Fig. 8. (Although the system extends to 1000—for the 28th letter—there is no need for letters beyond the nūn (50) in the astronomers' arithmetic, since no digit can be greater than 59. The only fact we need to add is that, as with the Greeks, "zero" was represented by ⱬ or ◌, which are two versions of the same cipher.)

ا	ب	ج	د	ه	و	ز	ح	ط
A	B	G	D	H	W	Z	Ḥ	Ṭ
1	2	3	4	5	6	7	8	9

ى	ك	ل	م	ن	س	ع	ف	ص
I	K	L	M	N	S	c	F	Ṣ
10	20	30	40	50	60	70	8ɔ	90

ق	ر	ش	ت	ث	خ	ذ	ض	ظ
Q	R	Sh	T	Th	Kh	Dh	Ḍ	Ẓ
100	200	300	400	500	600	700	8ȯ0	9ȯ0

غ
Gh
1000

Fig. 8

Thus, if we represent Arabic letters by the corresponding Latin ones in Fig. 8, the numeral 84 would be written *a kd* (i.e., $1 \cdot 60 + 24$), and *lb n* would represent $32\frac{50}{60}$. These two examples illustrate how the Muslims made a consistent adaptation of the Babylonian system to their own mode of writing, in the process of which they introduced a significant amount of cipherization. Of course, there remained the ambiguity of the value of any given numeral. Although *b mh* could represent 165 ($=2 \cdot 60 + 45$), it could equally well represent $2\frac{45}{60}$, and, in the absence of a sexagesimal point, some other device was necessary to eliminate this ambiguity.

There were two solutions to this problem. The one was to name each place, so that the nonnegative powers of 60 (1, 60, 60^2,...) were called "degrees," "first elevates," "second elevates,"..., while the negative powers of 60 (1/60, $1/60^2$, $1/60^3$,...) were called "minutes," "seconds," "thirds,".... The origin of the name "degrees" is in astronomy, where the term referred to the 360 equal parts into which the zodiac circle is divided. The term "minutes" is a translation of the *daqā'iq*, which means "fine," just as the English word "minute" does. The succeeding fine

Fig. 9

parts were, naturally, "the second, third, etc. fine parts." The other solution was to name the last place only, so that "*b mh minutes*" would make it clear that the value $2\frac{45}{60}$ was intended.

In the following survey of Muslim sexagesimal arithmetic we shall follow the second section of Kūshyār's *Principles of Hindu Reckoning,* and it is typical of the variety of approaches used by Muslim scientists that, although Kūshyār explains a consistent sexagesimal arithmetic, he does not use letters of the alphabet at all, but rather the form of the Hindu ciphers used in the Eastern caliphate. Thus, what some writers would express as *ka h mb*, Kūshyār writes as in Fig. 9, where the places of the numeral are written vertically in order to prevent confusion with the Hindu numeral 210,542. However, here as earlier he uses the ciphers only when he is actually showing the work. Elsewhere he writes out all the numerals longhand, and, to give some of the flavor of the work, we shall follow the same practice.

5.2 Sexagesimal Addition and Subtraction

To illustrate addition, Kūshyār gives the following example: "We wish to add twenty-five degrees, thirty-three minutes and twenty-four seconds to forty-eight degrees, thirty-five minutes and fifteen seconds." He sets these two numbers down in two columns, separated by an empty column, with degrees facing degrees, minutes facing minutes, and seconds facing seconds (Fig. 10). He then adds twenty-five to forty-eight, tens to tens, and units to units, and then he adds thirty-three to thirty-five and twenty-four to fifteen. Whenever a sum exceeds sixty he subtracts sixty from it, enters the result, and adds one to the place above it. This is the reason for the upper "one" shown in the second figure. The dust board where Kūshyār imagined these calculations being carried out would show the successive parts of Fig. 10, with only the last set of figures showing at the end.

Subtraction, too, is straightforward, and it proceeds from the highest place downwards, with borrowing when necessary. Figure 11 shows the process of subtracting rather than adding in the above example, and it clearly offers no serious difficulties.
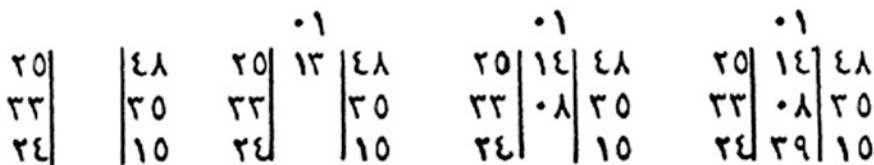
$$
\begin{array}{ccc}
& & •١ \\
٤٨ & ٢٥| & ٢٥| \\
٣٥ & ٣٣ & ٣٢ \\
١٥ & ٢٤ & ٢٤
\end{array}
\qquad
\begin{array}{ccc}
•١ & & \\
٤٨ & ١٢ & ٢٥| \\
٢٥ & & ٢٢ \\
١٥ & & ٢٤
\end{array}
\qquad
\begin{array}{ccc}
•١ & & \\
٤٨ & ١٤ & ٢٥| \\
٢٥ & •٨ & ٢٢ \\
١٥ & & ٢٤
\end{array}
\qquad
\begin{array}{ccc}
•١ & & \\
٤٨ & ١٤ & ٢٥| \\
٢٥ & •٨ & ٢٢ \\
١٥ & ٣٩ & ٢٤
\end{array}
$$

Fig. 10

$$
\begin{array}{ccc}
٤٨ & ٣٣ & ٢٥| \\
٢٥ & & ٢٢ \\
١٥ & & ٢٤
\end{array}
\qquad
\begin{array}{ccc}
٤٨ & ٣٣ & ٢٥| \\
٢٥ & •٢ & ٢٢ \\
١٥ & & ٢٤
\end{array}
\qquad
\begin{array}{ccc}
٤٨ & ٣٣ & ٢٥| \\
٢٥ & •١ & ٢٢ \\
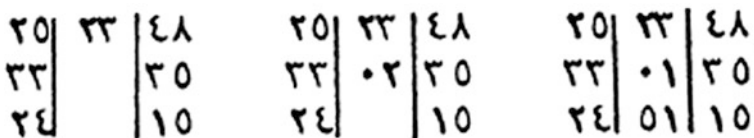١٥ & ٥١ & ٢٤
\end{array}
$$

Fig. 11

5.3   Sexagesimal Multiplication

5.3.1   Multiplication by Leveling

Multiplication and division were, however, another matter. Even such able mathematicians as al-Bīrūnī found it most convenient to convert the sexagesimal numerals to decimal form, perform the computations on the decimal forms by the rules of Hindu arithmetic, and then convert the answer back to sexagesimals, and the procedure was so common it was given a special name, "leveling". A contemporary of al-Bīrūnī, al-Nasawī, solves the problem of multiplying the two sexagesimal numbers $4°15'20''$ and $6°20'13''$ in the following way. First, he expresses both factors in terms of their lowest orders, Thus:

$$4°15' = (4 \cdot 60)' + 15' = 255', \text{ and } 255'20'' = (255 \cdot 60)'' + 20'' = 15,320''.$$

Similarly, he calculates the other factor to be $22,813.''$ Since the books that discuss this method explain how to calculate the products of various orders, al-Nasawī knows that the product of "seconds" by "seconds" will be on the order of "fourths" and, calculating in pure decimal numbers, he finds the product to be 349,495,160 fourths. Now it is necessary to perform the inverse operation of leveling, namely to "raise" this number to a sexagesimal expression, by dividing by 60. (We saw an example of this at the end of the treatment of division in the section on decimal arithmetic.) Thus, in this case,

$$349,495,160'''' = (5,824,919 \cdot 60 + 20)'''' = 5,824,919'''' + 20''''.$$

Finally, proceeding as above, but now with the thirds, then the seconds, and finally the minutes, al-Nasawī obtains the answer $26°58'1''59'''20''''$.

The foregoing, inelegant procedure was widespread but by no means universal. Kūshyār, although he mentions it in his treatise as one method, explains how to multiply two sexagesimal numbers without any such conversion.

5.3.2 Multiplication Tables

At the beginning of his section on sexagesimal arithmetic, Kūshyār describes a sexagesimal multiplication table, which consists of 59 columns, each headed by one of the integers from 1 to 59, and each containing 60 rows. The column headed with the integer 39, for example, contains in its rows the multiples of 39, from $1 \cdot 39$ to $60 \cdot 39$. Although Kūshyār's book has no such table, examples of these tables have survived in other treatises (See King and Plate 1.) The rightmost column of each page in such a table is headed "the number" and contains the alphabetic numerals, those from 1–30 usually appearing on the right-hand page and those from 31 to 60 on the left. The succeeding columns (going from right to left, as in Arabic writing) are headed by the alphabetic numerals between 1 and 60. (Of course only a certain number of them appear on each page, for reasons of space.) Each column gives, as we mentioned above, the first sixty multiples of the integer that stands at the top, and in general these multiples will need two sexagesimal digits to express them. For example, the product of 13 (*ig*) by 8 (*ḥ*) would be written with the two-digit numeral we transliterate as *a md*. The first twelve rows of the three rightmost columns in Plate 1, transliterated and then translated, are shown in Fig. 12. The eighth row below the heading, for example means that 8 13 = 1 44 (104) and 8 14 = 1 52 (112). (We use the convention that *n m*; *r s* means $n \times 60 + m + r/60 + s/60^2$. Another common convention separates the sexagesimal digits by commas, as *n, m; r, s*.)

A remarkable example of a multiplication table was compiled, probably by a Turkish astronomer, around the year 1600 and gives the first 60 multiples of each two-place sexagesimal number from 00 01 to 59 59, so one can find directly from the table such products as 14 34 · 19 = 4 36 46. The table has 212,400 entries and fills a ninety-page booklet. Other astronomers, doubtless, found it more convenient to use the more limited tables and compute other products as needed by one of the algorithms we shall now describe.

The first of these differs only slightly from the method Kūshyār gives for the multiplication of two decimal numbers. In the sexagesimal case, the numbers are written vertically rather than horizontally, with an empty column left between them to contain the product, and Kūshyār's procedure for the product of 25°42′ by 18°36′ is shown in Fig. 13.

Plate 1 Part of a sexagesimal multiplication table. The rightmost column is headed "the number" and shows the alphabetic numerals from 1 to 12. The succeeding columns (from right to left, as in Arabic handwriting) are headed by the numerals 13, 14, …, 18 and the entries underneath them give their multiples expressed as two-place sexagesimals. (See Fig. 12 for a transliteration and translation of the rightmost three columns of this table.) (Photo courtesy of the Egyptian National Library.)

ID	IG	Al-'adad	14	13	The Number
ID−	IG−	A	14 0	13 0	1
KH−	KW−	B	28 0	26 0	2
MB−	LT−	G	42 0	39 0	3
NW−	NB−	D	56 0	52 0	4
A I	A H	H	1 10	1 5	5
A KD	A IH	W	1 24	1 18	6
A LH	A LA	Z	1 38	1 31	7
A TB*	A MD	H	1 52	1 44	8
B W	A NZ	T	2 6	1 57	9
B K	B I	I	2 20	2 10	10
B LD	B KG	IA	2 34	2 23	11
B MH	B LW	IB	2 48	2 36	12

* An error for NB on the part of the scribe.

Fig. 12
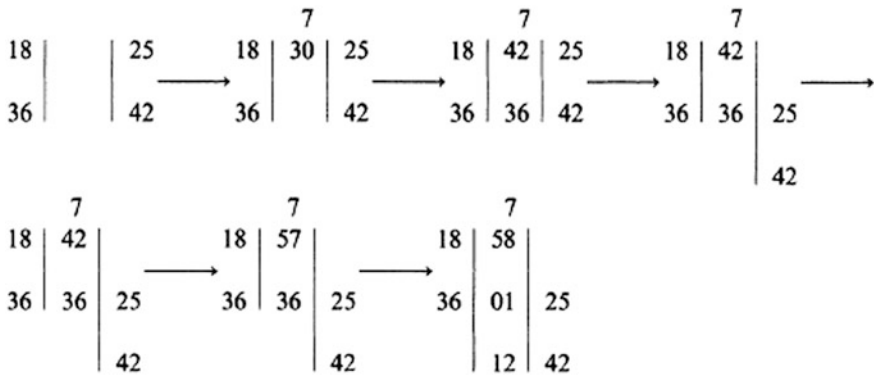
Fig. 13

5.3.3   Methods of Sexagesimal Multiplication

The first two steps, Kūshyār specifies, are done with the aid of the multiplication table for 18, and since the 30 in the first step and the 12 arising in the second are of the same order they must be added in the product, so 30 is replaced by 42.

Since the product of minutes by minutes is seconds, the answer is $7^{1}58°1'12''$ (that is, "7 first elevates, 58 degrees, 1 minute and 12 seconds"). In the last two steps, 36 is one place lower than 18 so its products with 25 and 42 must be added to the column where the answer is taking shape, but one place lower than the corresponding products for 18.
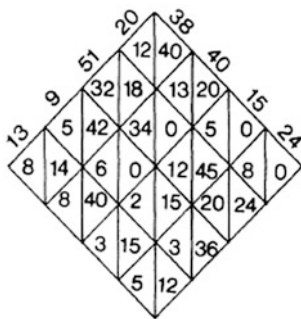
Fig. 14 The gelosia method of multiplication as described by al-Kāshī, Used for both decimal and sexagesimal systems it is shown here in the sexagesimal system. The answer is obtained by adding the entries in each of the eight columns within the square (base 60)

A method, that was popular both in Islam and the West for multiplication in a positional system, is illustrated in Fig. 14 with an example from Jamshīd al-Kāshī's *Calculators' Key*. The problem is to multiply 13 09 51 20 minutes by 38 40 15 24 thirds, and since the largest number of places is four a square is subdivided to form a lattice of 16 subsquares, each of which is divided as in Fig. 14 into two equal triangles. On the edges of the square that intersect at the top corner, the two factors are written so that the term of lowest order in one factor and that of highest order in the other factor are put at the top, each term of both factors being labeled by its orders. Then each square is filled in with the product of the two numbers on the outer edges opposite its sides, so that when this product has two ciphers the cipher of the highest order is put on the left of the square. For example, since 38 · 13 = 8 14 the 8 will be put in the left part of the left square and the 14 will be put in the right. When all 16 products are computed, the answer is obtained by adding up the ciphers in each of the eight vertical columns of the square, and the sums are written underneath. Since "minutes times thirds" is on the order of fourths, the lowest order of the product is fourths.

Although a certain amount of work is necessary to prepare the grid, it is then easy to fill in the lattice by means of a multiplication table, and the only computation involved is adding up the entries in the columns. Also, the squares can be filled in any convenient order, since the lattice-work keeps everything arranged.

The source from which we have taken this method, al-Kāshī's *The Calculators' Key*, gives no proof of the validity of the method; however, the proof is easy when one notices that what is put in the left-hand side of each square is precisely what would be carried and added to the next product in the method we are used to. The lattice does this carrying automatically, but what is carried is added to the product after all the multiplications have been done, rather than during the process as we are used to.

5.4 Sexagesimal Division

Finally, the method Kūshyār uses for division in the sexagesimal system parallels that used in the case of multiplication. Hence, to divide 49°36′ by 12°25′, Kūshyār arranges three columns and proceeds as in Fig. 15. Here

$$49 - 3 \cdot 12 = 13 \text{ and } 13\ 36 - 3 \cdot 25 = 13\ 36 - 1\ 15 = 12\ 21$$

so each digit of the divisor is multiplied by the digit of the quotient (obtained by trial, as we do), and the result is subtracted from that part of the dividend that is above it (or level with it) and to the right. Finally, the divisor is shifted down one place, and this is done so that the next digit of the quotient, when it is placed level with the highest entry of the divisor, will be of the correct order (in this case "minutes"). After the third step, the question now becomes "12°25′ times how many minutes produces something not exceeding 12°21′?", and the answer is "59 minutes." The last two steps of Fig. 15 show the final working-out. Again the general rule is that the product of a digit of the quotient by a digit of the divisor is subtracted from all of the dividend to the right and above (or level with) the digit of the quotient.

Thus Kūshyār gives the result as 3°59′ and says, "If we wish precision we copy the divisor one place lower." Hence, the result could be continued to as many sexagesimal places as necessary. Also, Kūshyār remarks that he has attached to his book a table giving "the results of the division," that is, the order that results when a number of one order (say "first elevates") is divided by that of another (say "thirds"). Finally, Kūshyār concludes his chapter with a discussion of how to calculate square roots in the sexagesimal system, an operation of some importance to astronomers.

Thus, there was widespread in the Muslim world a consistent system of sexagesimal arithmetic that permitted a unified treatment of both whole numbers and fractions. This system was supported by special tables, and it provided an approach to all the operations of arithmetic which was every bit as satisfactory as that of the (initially) less-developed system of decimal fractions.

Fig. 15

6   Square Roots

6.1   Introduction

Instead of following Kūshyār's presentation of the extraction of square roots, we shall follow that of Jamshīd al-Kāshī in his book *The Calculators' Key*, which we have already referred to as the work he wrote in Samarqand two years before his death in 1429. It is a compendium of elementary mathematics, including arithmetic, algebra, and the geometry of measurement, which contains a thorough treatment of decimal fractions, a table of binomial coefficients, and algorithms for extracting higher roots of numbers. For example, we shall see later how he works out the fifth root of a number on the order of trillions, namely 44,240,899,506,197.

The following list of titles of the five main chapters of *The Calculators' Key* shows its differences from the work of Kūshyār: (1) On the arithmetic of whole numbers. (2) On the arithmetic of fractions (including decimal fractions). (3) On the arithmetic of astronomers (sexagesimal). (4) On the measurement of plane and solid figures. (5) On the solution of problems by algebra.

6.2   Obtaining Approximate Square Roots

We shall first see how al-Kāshī extracts the square root of 331,781. His method for the square root is the same as Kūshyār's, but, unlike Kūshyār, al-Kāshī was writing for people who would use pen and paper. (It was in Samarqand where the Arabs first learned of papermaking from Chinese prisoners of war near the end of the eighth century and, because of its abundant supply of fresh water, Samarqand remained a center of paper manufacturing for several centuries.) Thus, in the method as al-Kāshī explains it, none of the intermediate steps are erased. Al-Kāshī organizes his work by dividing the digits of the radicand, 331,781, into groups of two called "cycles," starting from the right. (Thus 331,781 is divided as 33 17 81.) As al-Kāshī explains it, since the numbers 1, 100, 10,000 … have integer square roots (unlike 10, 1000, …), the cycles are relevant, for the first (81) counts the number of units, the second (17) the number of hundreds, the third (33) the number of 10,000's, etc. He then draws a line across the top of the radicand and lines down the paper separating the cycles. At the beginning, therefore, his paper looks like Fig. 16a.

To get the first of the three digits of his root he finds the largest digit n so that n^2 does not exceed 33. Since $5^2 = 25$ and $6^2 = 36$ he takes $n = 5$, which is written both above and some distance below the 33 (below the last 3) to obtain Fig. 16b.

Now he subtracts 25 from 33 to obtain 8, which he writes below 33, and draws a line under 33 to show he is done with it. (On the dust board the 33 would be erased and the 8 would replace it.) Now he doubles the part of the root he has obtained, 5, and writes the result (10) above the bottom 5, but shifted one place to the right to obtain Fig. 16c. (The dust board would only show the top 5, the middle 8 17 81,
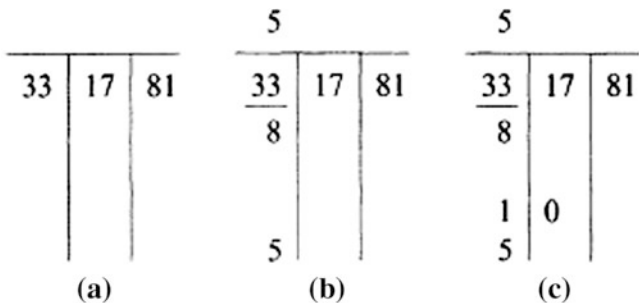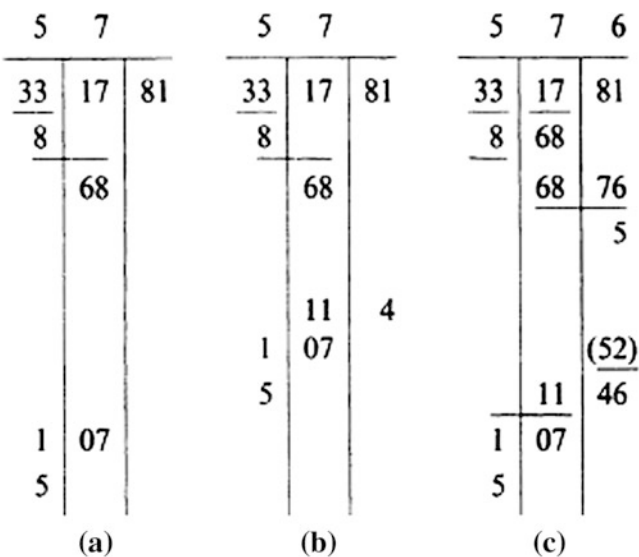
Fig. 16

Fig. 17

and the bottom 10 of Fig. 16c.) At this stage al-Kāshī has a current answer (5) at the top and double the current answer (10) at the bottom.

What al-Kāshī next asks is to find the largest digit x so that $(100 + x) \cdot x \leq 817$. Experiment shows $x = 7$, and he writes 7 above the 7 of 17, and next to the 10 on the bottom, and then he performs the computation of $(100 + 7) \cdot 7 = 749$ and subtracts the result from 817 to get Fig. 17a. He now begins the process again, doubling the last digit in 107 to get 114 and writing this above the 107, but shifted one place to the right, as shown in Fig. 17b. Once again he has the current answer (57) at the top and double that (114) at the bottom, and as before the question is this: What is the largest digit x so that $(1140 + x) \cdot x \leq 6881$? A trial division of

688 by 114 suggests trying $x = 6$. This works, and after $1146 \cdot 6$ has been sub-
tracted from 6881, the last digit of 1146 is doubled to make the 1146 into 1152
(=1146 + 6) as in Fig. 17c. (The dust board would show the top 576, the middle 5
and the bottom 1152.)

Thus al-Kāshī has obtained an approximate square root (576) as the current
answer and double that (1152) at the bottom. Finally, he increases 1152 by 1 to get
1153 and divides it into the remainder, the middle 5, to obtain, as the approximate
square root of 331,781, the number $576\frac{5}{1153}$ (=576.00434). Calculation shows the
square of the latter number is 331,780.996, so al-Kāshī's result is quite close.

6.3 Justifying the Approximation

Two questions arise: (1) What is the justification for al-Kāshī's procedure for
obtaining the integral part of the root; and (2) What is the justification for the
fractional part? We will begin with the second question.

6.3.1 Justifying the Fractional Part

In fact, the numerator of the fractional part, 5, is equal to $331{,}781 - (576)^2$, and the
denominator, 1153, is $577^2 - 576^2$. This is because 1153 is one more than "twice
the current answer", i.e.

$$1153 - 1 + 2 \cdot 576 = (1 + 576)^2 - 576^2 \,.$$

Thus the fractional part of the answer, $\frac{5}{1153}$, is just that obtained by *linear
interpolation*, i.e., $(331{,}781 - 576^2)/(577^2 - 576^2)$, a technique that was ancient
when Ptolemy used it in his *Almagest* in the first half of the second century A.D.

To understand this technique as a medieval astronomer might have justified it,
imagine a table of square roots obtained by listing in one column the successive
squares from 1^2 to $1{,}000^2$, and, next to these in a second column, the first thousand
whole numbers, as in Fig. 18.

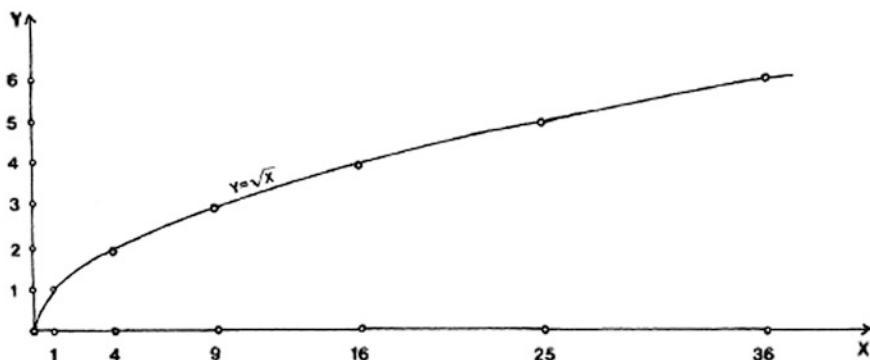| N	$\sqrt{N}$
1	1
4	2
9	3
331,776	576
332,929	577
1,000,000	1,000

Fig. 18

Fig. 19

Then, to find $\sqrt{6}$, the simplest procedure would be to observe that $4 < 6 < 9$ implies $2, < \sqrt{6} < 3$. Moreover, since $6 - 4 = 2$ and $9 - 4 = 5$, 6 is $\frac{2}{5}$ of the way between 9 and 4. Thus $\sqrt{6}$ is about $\frac{2}{5}$ of the way between $\sqrt{4} = 2$ and $\sqrt{9} = 2$, i.e., $\sqrt{6} = 2\frac{2}{5}$, approximately.

The reader will recognize that this reasoning is based on the assumption that $\sqrt{x}$ is proportional to x, which is the same as the assumption that, if we may express ourselves in modern language, the function $f(x) = \sqrt{x}$ is linear, i.e. its graph is a straight line. Although this is not true a glance at the graph of $f(x)$ in Fig. 19 reveals that it is nearly linear for $x > 1$ and over not too big an interval $[a, b]$. Thus, for example, the straight line joining the two points (16, 4) and (25, 5) is hardly distinguishable from the graph between these two points, and this is why the technique gives such a good approximation to the fractional part of $\sqrt{331,781}$. The table shows $576^2 = 331,776 < 331,781 < 332,929 = 577^2$ and, since $331,781 - 576^2 = 5$ while $577^2 - 576^2 = 1153$ we conclude that since $N = 331,781$ is $\frac{5}{1153}$ of the way between 576^2 and 577^2 its square root is about $\frac{5}{1153}$ of the way between 576 and 577. Thus $\sqrt{331,781} = 576\frac{5}{1153}$.

Linear interpolation was one of the standard ancient and medieval methods of what has been called "reading between the lines" (of tables), but we emphasize that the name "linear interpolation" reflects modern ideas, and those who discovered this method of approximation had no conception of a straight line as the graph of an equation. The ancient and medieval concept was simply that in a table of values pairing x to y one assumed that the change from y to y' was distributed equally over the units from x to x'.

6.3.2 Justifying the Integral Part

As for the extraction of the integral part of $\sqrt{N}$, al-Kāshī knows that if $N = abcdef$ then the largest integer r with the property that $r^2 \leq N$ has half as many digits as N,

in this case 3. (By taking $a = 0$ if necessary we may assume that N has an even number of digits.) Therefore he divides N into what he calls cycles, as $N = ab\ cd\ ef$ and thereby considers $N = ab \cdot 100^2 + cd \cdot 10^2 + ef$.

Now, al-Kāshī's first step is to find the largest number A so that $A^2 \le ab$. A will be a one-digit number, since ab has two digits and 10^2 has three digits. Such an A will be the first digit of the root, as the reader may verify.

His next step is to calculate the difference

$$\Delta_1 = N - (A \cdot 100)^2 = (ab - A^2) \cdot 100^2 + cd \cdot 10^2 + ef$$

and then to find the next place, i.e., the largest B so that

$$\Delta_2 = N - (A \cdot 100 + B \cdot 10)^2 \ge 0$$

He uses the basic identity $(X + Y)^2 = X^2 + (2X + Y)Y$ to expand

$$\Delta_2 \text{ as } N - (A \cdot 100)^2 - (2A \cdot 100 + B \cdot 10)B \cdot 10 = \Delta_1 - (2A \cdot 10 + B)B \cdot 100,$$

and the expression $2A \cdot 10 + B$ is the formal equivalent of al-Kāshī's instruction to double A, the previous digit of the root, and then put the digit (B) next to it. As al-Kāshī says, this next digit is chosen to be the largest, so that the product $(2A \cdot 10 + B)B \cdot 100$ does not exceed the previous difference Δ_1. The multiplication of $2A$ by 1000 instead of by 10,000 is reflected in its shift one place to the right. Of course, al-Kāshī never mentions the powers of 10 since they are automatically taken into account by the positioning.

The procedure should by now be clear. When we have determined B to be as large as possible so that $(A \cdot 100 + B \cdot 10)^2 \le N$ we choose C to be as large as possible   so   that $0 \le N - (A \cdot 100 + B \cdot 10 + C)^2$,   and   where,   with $X = (A \cdot 100 + B \cdot 10)$ and $Y = C$, $(X + Y)^2$ is again expanded according to the rule $(X + Y)^2 = X^2 + (2X + Y)Y$. This identity, or its alternate form $(X + Y)^2 - X^2 = (2X + Y)Y$ is the basis for the algorithm for the extraction of the square root. Al-Kāshī's procedure also takes advantage of the fact that in evaluating $N - (X + Y)^2$ the part $N - X^2$ has been evaluated at the previous step.

7   Al-Kāshī's Extraction of a Fifth Root

7.1   Introduction

We now follow the beginning of al-Kāshī's extraction of the fifth root of 44,240,899,506,197—a number on the order of trillions. The extraction of higher roots of numbers was, according to the testimony of ʿUmar Khayyām, an achievement of Muslim scholars, for he wrote in his *Algebra*,

From the Indians one has methods for obtaining square and cube roots, methods which are based on knowledge of individual cases, namely the knowledge of the squares of the nine digits 1^2, 2^2, 3^2 (etc.) and their respective products, i.e. $2 \cdot 3$ etc. We have written a treatise on the proof of the validity of those methods and that they satisfy the conditions. In addition we have increased their types, namely in the form of the determination of the fourth, fifth, sixth roots up to any desired degree. No one preceded us in this and those proofs are purely arithmetic, founded on the arithmetic of *The Elements*.

'Umar was neither the first mathematician nor the last who believed falsely that he was the originator of a method. In this case, we know that Abū al-Wafā', who flourished over 100 years before 'Umar, in the late tenth century, wrote a work entitled *On Obtaining Cube and Fourth Roots and Roots Composed of These Two.* Of course, 'Umar may not have known of Abū al-Wafā's treatise, or it may be that Abū a1-Wafā' simply pointed out that $\sqrt[4]{N} = (\sqrt{\sqrt{N}})$ and, since $\sqrt[3]{N}$ was already known from the Indians, roots such as the twelfth, for example, $\sqrt[12]{N} = \sqrt[4]{\sqrt[3]{N}}$ could be calculated by known methods. Thus Abū al-Wafā''s work may have been less innovative than that of 'Umar.

7.2 *Laying Out the Work*

However that may be, neither 'Umar's treatise nor that of Abū a1-Wafā' is extant, so we shall study al-Kāshī's method from Book III of his *Calculators' Key*. He begins by instructing the reader to write the number across the top of the page and to divide the number into cycles, which are, this time, successive groups of five digits beginning from the right. This is because the powers of 10 with perfect fifth roots are 1, 10^5, 10^{10}, etc. Next, al-Kāshī puts between the cycles, double lines and between the individual digits single lines, all running down the length of the page, and then he puts a line above the number, on which he will enter the digits of the root.

Next, he divides the space below the number into five broad bands by means of horizontal lines. The top band contains the number, and the words "Row of the number" are written on the edge of this band. The band below it is called "Row of the square square" (the fourth power) number." When this process is finished, the sheet will look as in Fig. 20, and everything is ready. It seems not too far from the algorithmic spirit of this procedure to look on the cells in Fig. 20 as locations in a computer's memory, and in keeping with this Fig. 21 shows a flow-chart for the root extraction which the reader may find useful to get an overview of the process.

7.3 *The Procedure for the First Two Digits*

Al-Kāshī' now proceeds as follows (Fig. 22). The largest integer, a, whose fifth power does not exceed 4424 is 5, so he puts 5 in "Row of Result" (above the first cycle) and at the bottom of "Row of Root." Next, he puts 5^2 (25) at the bottom of

Row of the result			
Row of the number	4 4 2 4	0 8 9 9 5	0 6 1 9 7
Row of the square–square			
Row of the cube			
Row of the square			
Row of the root			

Fig. 20

"Row of Square," 5^3 (125) at the bottom of "Row of Cube," and 5^4 (625) at the bottom of the "Row of the Square Square". Finally, $4424 - 5^5 = 1299$ is placed in the "Row of Number" (This number, in virtue of its position, represents $1299 \cdot 10^{10}$.)

Next he begins the process called "once up to the row of the square square," by adding the latest entry in "Row of Root" (5) to the most recently obtained digit of the root (5) and writing the sum (10) in "Row of Root" above 5. Now, he multiplies the sum by 5 and puts the product, $10 \cdot 5$, above 5^2 in "Row of Square" and then adds the two to get $75 = 5^2 + 50$. The sum he multiplies by 5 and puts the product $(75 \cdot 5)$ above 5^3 in "Row of Cube." Again, he adds these to get $500 = 5^3 + 75 \cdot 5$. Then he multiplies the sum by 5 and puts $500 \cdot 5$ above 5^4 in "Row of Square Square." Finally, he adds these to get $3125 = 5^4 + 500 \cdot 5$. (The lines within the bands mean, in the case of the bottom four bands, that all numbers below them would be erased on a dust board, and, in the case of the top band, the numbers *above* would be erased.)

Now, beginning with the 10 in "Row of Root," he repeats the above as far as "Row of Cube" ($10 + 5 = 15$, etc.), then, with 15, to "Row of Square" ($15 + 5 = 20$, etc.) and finally $20 + 5 = 25$ is put in "Row of Root."

Thus the numbers lying entirely in the first column are obtained. Now 3125 (in "Row of Square Square") is shifted one place right, 1250 (in "Row of Cube") two places, 250 (in "Row of Square") three places, and finally 25 (in "Row of Root") four places to the right, and he puts this "25" at the bottom of the next column (below the cycle 08995), as in Fig. 22.

At this point he seeks b, the largest single digit so that $f(b) \leq 129,908,995 = D$, where

$$f(b) = b(((b \cdot 25b + 250 \cdot 10^2)b + 1250 \cdot 10^3)b + 3125 \cdot 10^4)$$

where "$25b$" means $250 + b$.

It turns out that $f(4) = 146,665,024$ is too big, and since $f(3) = 105,695,493 < D$), al-Kāshī concludes that 3 is the desired value for b. (This method of evaluating a polynomial is standard in numerical analysis and is called Horner's method in many texts on the subject.)
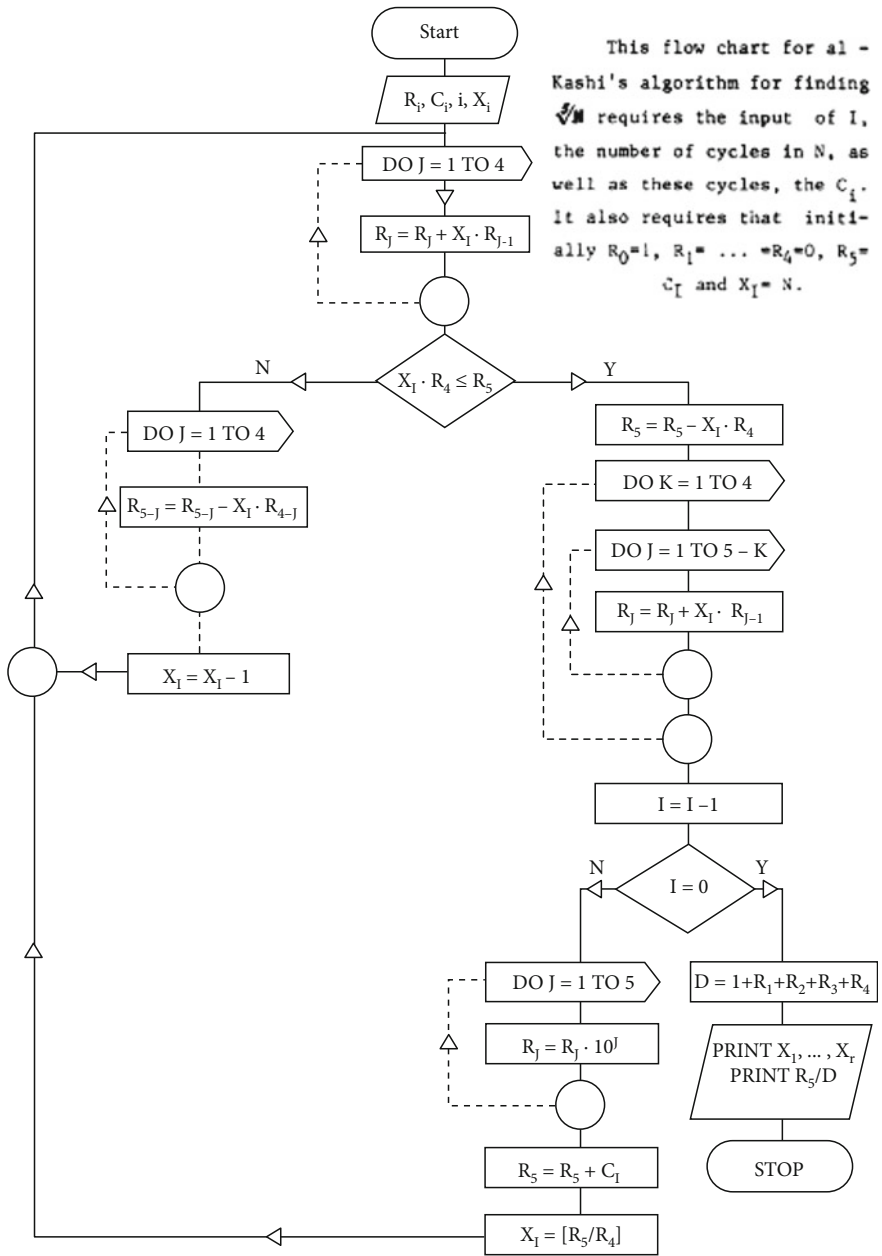
Start

R_i, C_i, i, X_i

DO J = 1 TO 4

$R_J = R_J + X_I \cdot R_{J-1}$

This flow chart for al -
Kashi's algorithm for finding
$\sqrt[5]{N}$ requires the input of I,
the number of cycles in N, as
well as these cycles, the C_i.
It also requires that initi-
ally $R_0 = 1$, $R_1 = \ldots = R_4 = 0$, $R_5 =$
C_I and $X_I = N$.

$X_I \cdot R_4 \leq R_5$

N   Y

DO J = 1 TO 4 $R_5 = R_5 - X_I \cdot R_4$

$R_{5-J} = R_{5-J} - X_I \cdot R_{4-J}$ DO K = 1 TO 4

DO J = 1 TO 5 – K

$R_J = R_J + X_I \cdot R_{J-1}$

$X_I = X_I - 1$

I = I – 1

N   Y

I = 0

DO J = 1 TO 5 $D = 1 + R_1 + R_2 + R_3 + R_4$

$R_J = R_J \cdot 10^J$

PRINT $X_1, \ldots, X_r$
PRINT R_5/D

$R_5 = R_5 + C_I$

STOP

$X_I = [R_5/R_4]$

Fig. 21

Row of the result	5		
Row of the number	4 4 2 4 3 1 2 5 1 2 9 9 = D	0 8 9 9 5	0 6 1 9 7
Row of the square–square	3 1 2 3 1 2 5 → 2 5 0 0 6 2 5	5 Last step up to Row of Square Square	
Row of the cube	1 2 1 2 5 0 7 5 0 5 0 0 → 3 7 5 1 2 5	5 0 Third step up to Row of Square Square	
Row of the square	2 5 0 1 0 0 1 5 0 7 5 7 5 → 5 0 2 5	2 5 0 Second step up to Row of Square Square	
Row of the root	2 5 2 0 1 5 1 0 → 5	 Start "Once up to Row of Square Square" 2 5	

Fig. 22

7.4 *Justification for the Procedure*

The reader may easily verify with a pocket calculator that $530^5 < N$, while $540^5 > N$, and thus al-Kāshī has found the next digit of the fifth root. The question is: "How?," and the answer lies in the analog of the identity that underlies extraction of square roots. If $C(n, k)$ denotes the binomial coefficient "n choose k,"

which counts the number of ways of choosing k objects from a set of n objects, then the binomial theorem applied to squares may be written

$$(A+B)^2 - A^2 = (C(2,2)B + C(2,1)A)B$$

and, applied to higher powers, yields the identities:

$$(A+B)^3 - A^3 = ((C(3,3)B + C(3,2)A)B + C(3,1)A^2)B,$$
$$(A+B)^4 - A^4 = (((C(4,4)B + C(4,3)A)B + C(4,2)A^2)B$$
$$+ C(4,1)A^3)B,$$
$$(A+B)^5 - A^5 = ((((C(5,5)B + C(5,4)A)B + C(5,3)A^2)B$$
$$+ C(5,2)A^3)B + C(5,1)A^4)B.$$

The numbers $C(n, k)$ are arranged in a triangular array in Fig. 23. Notice that each row of this triangle begins and ends with a I and that a number greater than 1 in any row is just the sum of the two numbers to the right and left of it in the row above it. (Thus in the fourth row the "3" is the sum of 1 and 2 in the row above it.) If we begin numbering the rows with 0 and use the convention $C(0, 0) = 1$, then for all $0 \leq k \leq n$, $C(n, k)$ is the kth entry in Row n, and the rule for generating the triangle corresponds to the fundamental relationship

$$C(n,k) = (n-1,k) + C(n-1,k-1)$$

This triangle is called "Pascal's Triangle," after the French mathematician of the early seventeenth century, Blaise Pascal, whose *Traité du Triangle Arithmétique*, published in 1665, drew the attention of mathematicians to its properties. However, it might with more justice be called al-Karajī's triangle, for it was al-Karajī who, around the year A.D. 1000, drew the attention of mathematicians in the Islamic world to the remarkable properties of the triangular array of numbers.

If we substitute the values of $C(5, k)$ into the expression for $(A + B)^5 - A^5$ we obtain the equality

$$(A+B)^5 - A^5 = ((((B+5A)B + 10A^2)B + 10A^3)B + 5A^4)B.$$

```
                 1
              1     1
           1     2     1
        1     3     3     1
     1     4     6     4     1
  1     5    10    10     5     1
1     6    15    20    15     6     1
1  7    21    35    35    21    7    1
```
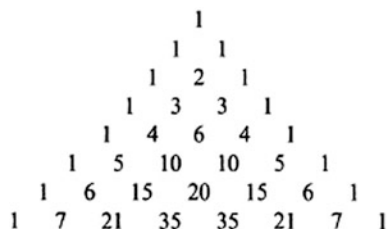
Fig. 23

In the present case, $A = 5 \cdot 10^2$ and $B = b \cdot 10$, and if we substitute these values for A and B the right-hand side of the expression now becomes

$$10^5((((b + \mathbf{25} \cdot 10)b + \mathbf{250} \cdot 10^2)b + \mathbf{1250} \cdot 10^3)b + \mathbf{3125} \cdot 10^4)b,$$

and the numbers in boldface are those appearing in the function f given earlier.

To see how al-Kāshī's technique generates the binomial coefficients, begin with a page divided into four horizontal bands, and, instead of writing the entries within a band one above the other, write them in a row, towards the right. Now, fill in the page as follows:

1. Put the first four powers of 1 up the leftmost column, one in each band.
2. If any column has been filled in, start the next at the bottom by adding 1 to the entry to the left of it.
3. If any column has been filled in up to a given row, fill in the next row of that column by adding 1 times the entry in the given row to that in the previous column of the next row.
4. Each column after the second contains one less row than the column to its left.

These rules will generate Fig. 24 in which the columns are just the diagonals descending to the right in Pascal's triangle, apart from the initial "1"s in these diagonals.

Of course, al-Kāshī wants not just the binomial coefficients $C(5, k)$, but the values $5C(5, 4) = 25$, $5^2 C(5, 3) = 250$, $5^3 C(5, 2) = 1250$, and $5^4 C(5, 1) = 3125$. Thus, we construct a figure, on the model of Fig. 24, but this time:

1. Put ascending powers of 5 up the first column.
2. Add 5 instead of 1 as we move across the bottom row.
3. Whenever we move a number up to add it, first multiply it by 5.

Then we obtain Fig. 25 in which the last entries of the rows are the coefficients given in boldface in the expansion of f earlier.

One point remains, however. The numbers al-Kāshī must calculate with are not quite the above but $25 \cdot 10^6$, $250 \cdot 10^7$, $1250 \cdot 10^8$, and $3125 \cdot 10^9$. To represent these numbers on the array, al-Kāshī must move the 25 to the right four spaces, the

```
1   5
1   4   10
1   3   6    10
1   2   3    4    5
```

Fig. 24

```
625   3125
125    500   1250
 25     75    150   250
  5     10     15    20    25
```

Fig. 25

250 three spaces, the 1250 two, and the 3125 one. This is because, where they are, they are being treated as if they were multiplied by 10^{10}, while, in $f(b)$, 25 is multiplied only by 10^6. Since $10 - 6 = 4$, al-Kāshī must shift the 25 four places to the right to make it represent $25 \cdot 10^6$, etc. This is sufficient explanation of why, after al-Kāshī has found b according to the procedure outlined and has subtracted $f(b)$ from D, there remains $D' = N - (A + B)^5$.

7.5   The Remaining Procedure

Figure 26 shows the next part of the algorithm after 3 has been placed both in "Row of Result" and next to 25 in "Row of Root" (to form the number 253). The numbers in the parentheses on the right show how the algorithm calculates $f(b)$ in stages. Thus, 253 is multiplied by 3 to obtain 759, which is then put directly above the 25000 (the last two zeros not being shown) and added to it to obtain 25,759. This is then multiplied by 3, written above the 1,250,000, and added to obtain 1,327,277. Finally, this is multiplied by 3 and the product added to the 31,250,000 in the row above it. This sum is finally multiplied by three and the product, which is $f(3)$, is subtracted from 129,908,995 in the "Row of the number." The difference, D', is $D - f(3)$.

   Next, al-Kāshī begins the process "once up to the row of the square square" (with $253 + 3 = 256$, etc.), then to the row of the cube (with $256 + 3 = 259$), then to the row of the square ($259 + 3 = 262$). Finally, in the row of the number he puts $265 = 262 + 3$. The multiplications of course are all by 3 instead of by 5. The top numbers in the bands are then shifted so the numbers obtained will represent the constants in the polynomial:

$$g(c) = (((c \cdot 265c + 28,090 \cdot 10^2)c + 1,488,770 \cdot 10^3)c + 39,452,405 \cdot 10^4)c,$$

   The next digit, c, must satisfy a condition entirely analogous to the one b satisfied, i.e., it must be the largest single digit so that $g(c)$ does not exceed $24,213,502 \cdot 10^5$. Al-Kāshī finds $c = 6$.

   The bracketed lines in Fig. 27 denote the computation of the terms of $g(6)$, and D'' the final difference. Finally, al-Kāshī performs the procedure of going up to the "Row of Square Square," etc. The reader should now be able to follow without difficulty the steps as shown in Fig. 27.

7.6   The Fractional Part of the Root

At this point al-Kāshī has finished the calculation of the integer part of the fifth root of the given 14-place number. He had perfect control of decimal fractions and there is no doubt he knew that he could now shift again and continues the procedure to

Row of the result	5	3	
Row of the number	1299 1056 242	08995 95493 = f(3) 13502 = D'	06197
Row of the square–square	39 394 42 352 39 312	45240 52405 20574 31831⎤ 81831⎬Fourth step in computing f(3) 5⎦	5
Row of the cube	 14 14 13 12	14887 88770 81912 06858 79581 27277⎤ 77277⎬Third step in computing f(3) 50⎦	70
Row of the square		28 28090 786 27304 777 26527 768 25759⎤ 759⎬Second step in computing f(3) 250⎦	090
Row of the root		265 262 9 6 253}Begin computation of f(3)	265

Fig. 26

extract successive decimal places of the fifth root. Also, al-Khwārizmī, in a part of his treatise on arithmetic reported by the Latin writer John of Seville (fl. ca. 1140), gives an example of calculating $\sqrt{2}$ by calculating:

Row of the result	5	3	6	
Row of the number	242	13502	06197	
	{242	13502	06176	
			21	= D″
Row of the square–square	41	26949	58080 → Last step to square square	
		91365	90384	
	[40	35583	67696 → Sum of two rows below	
		90343	17696 ←	
	[39	45240	5	×6
Row of the cube		15399	06560 → Last step to cube	
		171	41496	
		15227	65064 → Third step to square square	
		170	45448	
	[15057	19616 → Sum of two rows below		
	169	49616 ←		
	[14887	70	×6	
Row of the square		28	72960	
		16044		
		28	56916 → Second step to row of cube	
		16008		
		28	40908 → Second step to square square	
		15972		
	[28	24936 → Sum of two rows below		
	15936 ←			
	[28	090		
Row of the root			2680	×6
			2674	
			2668 → First step to row of cube	
			2662 → First step to row of square square	
			{2656 → Begin computation of g(c)	

Fig. 27

$$\sqrt{2} = \frac{\sqrt{2,000,000}}{1000} \doteq \frac{1414}{1000}$$

a procedure which al-Kāshī also recommends. So, even without decimal fractions, one can obtain any desired degree of accuracy.

What al-Kāshī does here, however, is to add up the top numbers in each of the four bottom rows and increase the sum by one, i.e., he forms

$$412694958080$$
$$1539906560$$
$$2872960$$
$$2680$$
$$+ \qquad\qquad 1$$
$$\overline{414237740281}$$

and states that the fifth root of the given number is 536 + 21/414,237,740,281.

Al-Kāshī's rule for finding the fractional part is based on the approximation

$$\left(n^k + r\right)^{1/k} = n + \frac{r}{(n+1)^k - n^k}$$

where he explicitly calculates

$$(n+r)^k - n^k = C(n,1)n^{k-1} + C(n,2)n^{k-2} + \ldots + 1.$$

This is, of course, just the method of linear interpolation we discussed earlier, for $n^k + r$ is r units of the total $(n+1)^k - n^k$ units between two successive kth powers, so linear interpolation would place its kth root, $(n^k + r)^{1/k}$, the fraction $r/[(n+1)^k - n^k]$ of the way between the two kth roots n and $n + 1$.

Figure 28 reproduces a page from the printed version of al-Kāshī's *The Calculators' Key*, which shows the entire calculation we have just explained. The reader will benefit from identifying the numerals and following the procedure through the first "Once up to the row of square square."

8   The Islamic Dimension: Problems of Inheritance

Al-Khwārizmī devotes the first half of his book on algebra to solutions of the various types of equations and demonstrations of the validity of his methods, but the latter half contains examples of how the sciences of arithmetic and algebra could be applied to the problems posed by the requirements of the Muslim laws of inheritance.

When a person dies who leaves no legacy to a stranger the calculation of the legal shares of the natural heirs could be solved by the arithmetic of fractions. The calculation of these shares was known as '*ilm al-farā'iḍ*, and two examples from al-Khwārizmī's work illustrate the applications of arithmetic here.

Fig. 28

8.1   The First Problem of Inheritance

This problem is a simple one, namely,

Example 1 "A woman dies, leaving her husband, a son and three daughters," and the object is to calculate the fraction of her estate that each heir will receive.

The law in this case is that the husband receives $\frac{1}{4}$ of the estate and that a son receives twice as much as a daughter. (It should be said, however, that from the woman's point of view Islamic inheritance law was a considerable improvement over what the pre-Islamic requirements in the Arabian Peninsula had been.)

Al-Khwārizmī then divides the remainder of the estate after the husband's share has been deducted, namely $\frac{3}{4}$, into five parts, two for the son and three for the daughters. Since the least common multiple of five and four is twenty, the estate should be divided into twenty equal parts. Of these, the husband gets five, the son six, and each daughter three.

8.2   The Second Problem of Inheritance

This problem is a little more complicated and illustrates how unit fractions were employed to describe more complex fractions.

Example 2 A woman dies, leaving her husband, son and three daughters, but she also bequeaths to a stranger $\frac{1}{8} + \frac{1}{7}$ of her estate. Calculate the shares of each.

One law on legacies is that a legacy cannot exceed $\frac{1}{3}$ of the estate unless the natural heirs agree to it. (Here complications could enter because of the provision that if some agree and some do not agree those who do agree must pay, pro-rated, their share of the excess of the legacy over the third.) In the present case, however, since $\frac{1}{8} + \frac{1}{7} \leq \frac{1}{3}$ no complications enter, and the second provision on legacies, namely that a legacy must be paid before the other shares are calculated, now takes effect.

As in the above problem, the common denominator of the legal shares of her relatives is 20. Also, the fraction of the estate remaining after the stranger's legacy $\left(\frac{1}{8} + \frac{1}{7} = \frac{15}{56}\right)$ has been paid is $\frac{41}{56}$ Then the ratio of the stranger's share to the total shares of the family is $\left(\frac{15}{56}\right) : \left(\frac{41}{56}\right) = 15 : 41$. Thus, of the whole estate, the stranger will receive 15 parts to the 41 parts the natural heirs will receive. Multiplying both numbers by 20 to facilitate the computation of the shares of the heirs, we find that of a total of 20 (15 + 41) = 20 · 56 = 1120 parts the stranger receives 20 · 15 = 300 and the heirs jointly receive 20 · 41 = 820. Of these parts, the husband receives $\frac{1}{4}$, namely 205, the son $\frac{6}{20}$ namely 246, and each of the daughters gets 123.

8.3   *On the Calculation of* **Zakāt**

Another example of the use of arithmetic in the Islamic faith is in the calculation of *zakāt*, the community's share of private wealth. This is payable each year, at a certain rate, and the following problem, taken from *The Supplement of Arithmetic* of the eleventh-century mathematician, Abū Manṣūr al-Baghdādī, follows the gradual diminution of a sum of money as the *zakāt* is paid for 3 years. Its treatment of the fractional parts of a *dirhām* reminds one of Kūshyār's treatment of fractions, and in presenting it we paraphrase slightly, following the translation in Saidan (1987) "We want to pay the *zakāt* on 7586 *dirhāms*, the amount that Muḥammad ibn Mūsā al-Khwārizmī mentioned in his work." (The *dirhām* was divided into sixty *fulūs*, the plural of *fils* (see Plate 2).)

The rate of *zakāt* is 1 *dirhām* in 40, but al-Baghdādī does not divide 7586 by 40 according to the algorithm Kūshyār describes. Rather he calculates the total due on 7586 *dirhāms*, place-by-place, as follows:

From the first place we remove 1, which we make 40, and then remove 6 from the 40. This 6 is the *zakāt* due on 6 *dirhāms* and it is 6 parts of (the 40 into which we have divided) a *dirhām*. Thus, of the 40 there remains 34 parts. This we put under the five that has remained in the units place, as in Fig. 29a.

We must now calculate $\frac{1}{40}$ of the 80 that arises from the 10's place, to obtain 2, which we subtract from the five in the unit's place. This leaves what is shown in Fig. 29b.

In the 100's place there is 500, on which the *zakāt* due is $12\frac{1}{2}$. Of the 40 parts into which we have divided the *dirhām*, $\frac{1}{2}$ is 20, so when we subtract this from 34 there remain 14 parts. Also 12 from 83 leaves 71, so there now remain the figures shown in Fig. 29c.

Finally, $\frac{1}{40}$ of the 7000 we obtain from the 1000's place is 175, and when we subtract this from 571 there remains 396, so the answer is that shown in Fig. 29d.

Al-Baghdādī follows this for two more years, after which there remain the number of *dirhāms* shown in Fig. 29e, where, e.g. the 14 means $14/(40)^3$ *dirhāms*. (The tax collector is going to get every last *fils* due!)

Of course, *dirhāms* are divided into 60 *fulūs*, not 40, and so, to calculate the *zakāt*, the base-40 fractions, which were convenient to use in the intial stage, must now be converted into sexagesimal fractions, and here al-Baghdādī points out a slip on the part of al-Khwārizmī, his source for the problem. Evidently, al-Khwārizmī said that if each of the fractional parts (i.e., 6, 8 and 14) is increased by $\frac{1}{2}$ then they become sexagesimal parts, i.e., minutes, seconds and thirds. This is of course true for the 6, because

$$\frac{6}{40} = 6 \cdot \frac{\frac{3}{2}}{40 \cdot \frac{3}{2}} = \frac{9}{60}$$

but it is false for the following parts, and al-Baghdādī gives the correct rule.

Plate 2 Obverse and reverse of two coins from the medieval Islamic world. The one on the right is a *fils* of Damascus, minted in 87 A.H. (Anno Hijra). The obverse of the coin on the right names the Caliph al-Walīd (of the Umayyad Dynasty) and gives the *shahāda* (Muslim confession of faith: "There is no god but Allah and Muḥammad is the Messenger of Allāh.") The one on the left is a *dirham* of Medīnat al-Salām (Baghdad) issued in 334 A.H. and names, on the two faces of the coin, the Būyid rulers Muʿizz al-Dawla and ʿImād al-Dawla as well as the Caliph al-Muṭīʿ. (Photo courtesy of the American Numismatic Society, New York.)

	(a)	(b)	(c)	(d)	(e)
7 5 8 6	7 5 8 5	7 5 8 3	7 5 7 1	7 3 9 6	7 0 3 1
	34	34	14	14	6
					8
					14

Fig. 29

Without question, ʿ*ilm al-farāʾiḍ* is an important subject for Muslims, but in estimating the place of mathematics in that discipline a cautionary note, written by the great fourteenth century Maghribi historian, Ibn Khaldūn, is worth recording.

> Religious scholars in the Muslim cities have paid much attention to it. Some authors are inclined to exaggerate the mathematical side of the discipline and to pose problems requiring for their solution various branches of arithmetic, such as algebra, the use of roots, and similar things. It is of no practical use in inheritance matters because it deals with unusual and rare cases. (Transl. in Rosenthal, cited in bibliography of Chap. 1.)

Exercises

1. Use Kūshyār's method to add and subtract 12,431 and 987, showing your steps as in the text.
2. Develop an algorithm for halving a number that starts with the highest place in a number. Why do you think the Muslim calculators worked from the lowest place?
3. Use an operation modeled on raising to obtain a *decimal* expansion of $\frac{243}{7}$.
4. Use Kūshyār's method to multiply 46 by 243.
5. Use Kūshyār's method of division to divide 243 by 7, and then use the method of raising to find a 3-place sexagesimal approximation to $\frac{5}{7}$.
6. Adapt the method of raising to find a 3-place decimal approximation to $\frac{5}{7}$.
7. Devise a method for converting decimal integers to sexagesimal integers. Do the same for fractions. Now do the latter, but going from sexagesimal to decimal.
8. List some possible values for *ke mb h*, including some fractional ones.
9. Add, subtract and multiply the two sexagesimal numbers 36, 24 and 15, 45. Divide 2, 6, 15, 0 by 8, 20.
10. Use the lattice method to multiply 2468 by 9753.
11. Use the procedure in Sect. 3 to express 19/35 as a related fraction.
12. With A and N as in the section on square roots show that $((A + 1) \cdot 100)^2 > N$, while $(A \cdot 100)^2 < N$. Conclude that A is the first digit of the root.
13. Use al-Kāshī's method, including linear interpolation, to find $\sqrt{20,000}$.
14. If a man dies, leaving no children, then his mother receives $\frac{1}{6}$ and his widow $\frac{1}{4}$ of the estate. If he has any brothers or sisters, a brother's share is twice that of a sister. Find the fractions of the estate due if a man dies, leaving no children but a wife, a mother, a brother, two sisters, and a legacy of $\frac{1}{9}$ of the estate to a stranger.
15. Give a rule for converting the remaining base-40 parts in the example from al-Baghdādī to sexagesimal parts. Generalize this rule to one for converting fractions from base *n* to base *m*.

16. Show that for any single digit b

$$f(b) = (5 \cdot 10^2 + b \cdot 10)^5 - (5 - 10^2)^5$$

and conclude that b is the desired second digit of the fifth root, where f is the function in our discussion of al-Kāshī's extraction of the fifth root.

17. Al-Kāshī's method of evaluating $f(b)$ suggests evaluating an arbitrary polynomial

$$g(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

as

$$g(x) = (\ldots(a_n x + a_{n-1})x + a_{n-2})x + \cdots + a_1)x + a_0,$$

where the initial dots denote an appropriate number of parentheses and those in the middle denote intermediate terms.

(a) Evaluate $g(2)$, where $g(x) = 5x^3 - 3x^2 + 7x + 6$ by this method.
(b) If addition and multiplication are each counted as one operation how many operations are necessary to evaluate $g(x)$ by this formula? How many are necessary according to the usual method?

18. Show that the sum $412{,}694{,}958{,}080 + \cdots + 1$ calculated in al-Kāshī's extraction of the fifth root is equal to $537^5 - 536^5$.

19. Use al-Baghdādī's method and format (as in Fig. 29) to supply the details of the computation of the *zakāt* for year two.

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